CRITERIA FOR THE BOUNDEDNESS OF POTENTIAL OPERATORS IN GRAND LEBESGUE SPACES

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Abstract. It is shown that that the fractional integral operators with the parameter α, $0<\alpha<1$, are not bounded between the generalized grand Lebesgue spaces $L^{p),\theta_1}$ and $L^{q),\theta_2}$ for $\theta_2 < (1+\alpha q)\theta_1$, where $1<p<1/\alpha$ and $q=\frac{p}{1-\alpha p}$. Besides this, it is proved that the one–weight inequality

$$\|I_\alpha(fw^\alpha)\|_{L^{q),\theta(1+\alpha q)}_w} \le c\|f\|_{L^{p),\theta}_w},$$

where I_α is the Riesz potential operator on the interval $[0,1]$, holds if and only if $w \in A_{1+q/p'}$.

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Introduction

In this paper we show that potential operators with the parameter α, $0<\alpha<1$, are not bounded from $L^{p)}$ to $L^{q)}$, where $1<p<\infty$ and q is the Hardy–Littlewood–Sobolev exponent of p: $q=\frac{p}{1-\alpha p}$. This phenomena motivates us to investigate the boundedness problem for the Riesz potential operator I_α in the generalized grand Lebesgue spaces. In particular, we study this problem in $L^{p,\theta}_w$ spaces and prove that the one–weight inequality

$$\|I_\alpha(fw^\alpha)\|_{L^{q),\theta(1+\alpha q)}_w([0,1])} \le c\|f\|_{L^{p),\theta}_w([0,1])}$$

holds if and only if w belongs to the Muckenhoupt's class $A_{1+q/p'}$.

The unweight spaces $L^{p,\theta}$ (i.e. $L^{p,\theta}_w$ for $w \equiv const$) were introduced by E. Greco, T. Iwaniec and C. Sbordone [6] when they studied existence and uniqueness of the nonhomogeneous $n-$ harmonic equation $divA(x, \nabla u) = \mu$.

The grand Lebesgue spaces $L^{p)} = L^{p),1}$ first appeared in the paper by T. Iwaniec and C. Sbordone [7]. In that paper the authors showed that if $f = (f_1, \cdots, f_n) : \Omega \to \mathbb{R}^n$ belongs to the Sobolev class $W^{1,1}$, where Ω is an open subset in $\mathbb{R}^n$, $n \geq 2$, then the Jacobian determinant $J = J(f, x) = det\ Df(x)$ ($J(x, f) \geq 0$ a.e.) of f belongs to the class $L^1_{loc}(\Omega)$ provided that $g \in L^{n)}$, where

$$g(x) := |Df(x)| = \{\sup |Df(x)y| : y \in S^{n-1}\}.$$

Recently necessary and sufficient conditions guaranteeing the one–weight inequality for the Hardy–Littlewood maximal operator in $L^{p)}_w(I)$, where $I = [0, 1]$, were established by A. Fiorenza, B. Gupra and P. Jain [4], while the same problem for the Hilbert transform was studied in the paper [8]. In particular, it turned out that the Hardy–Littlewood maximal operator (resp. the Hilbert transform) is bounded in $L^{p)}_w(I)$ if and only if the weight w belongs to the Muckenhoupt class $A_p(I)$.

1 Preliminaries

Let Ω be a bounded subset of $\mathbb{R}^n$ and let w be an a.e. positive, integrable function on Ω (i.e. a weight). The weighted generalized grand Lebesgue space $L^{p),\theta}(\Omega)$ $(1 < p < \infty)$ is the class of those $f : \Omega \to \mathbb{R}$ for which the norm

$$\|f\|_{L^{p),\theta}_w(\Omega)} = \sup_{0<\varepsilon<p-1} \left(\frac{\varepsilon^\theta}{|\Omega|} \int_\Omega |f(t)|^{p-\varepsilon} w(t) dt \right)^{1/(p-\varepsilon)}$$

is finite.

If $w \equiv 1$, then we denote $L^{p),\theta}(\Omega) := L^{p),\theta}_1(\Omega)$. The space $L^{p),\theta}_w(\Omega)$ is not rearrangement invariant unless $w \equiv\ const$.

Hölder's inequality and simple estimates yield the following embeddings (see also [6], [4]):

$$L^p_w(\Omega) \subset L^{p),\theta_1}_w(\Omega) \subset L^{p),\theta_2}_w(\Omega) \subset L^{p-\varepsilon}_w(\Omega), \tag{1.1}$$

where $0 < \varepsilon < p - 1$ and $\theta_1 < \theta_2$.

In the classical weighted Lebesgue spaces L^p_w the equality

$$\|f\|_{L^p_w} = \|w^{1/p} f\|_{L^p}$$

holds but this property fails in the case of grand Lebesgue spaces. In particular, there is $f \in L^{p)}_w$ such that $w^{1/p} f \notin L^{p)}$ (see also [4] for the details).

Let φ be positive increasing function on $(0, p-1)$ satisfying the condition $\varphi(0+) = 0$, where $1 < p < \infty$. We will also need the following auxiliary class of functions defined on Ω and associated with φ:

$$L_w^{p),\varphi(x)}(\Omega) := \left\{ f : \sup_{0<\varepsilon\leq p-1} \left(\varphi(\varepsilon)^{\frac{1}{p-\varepsilon}} \|f\|_{L_w^{p-\varepsilon}} \right) < \infty \right\}.$$

The space $L_w^{p),\theta}(\Omega)$, $\theta > 0$, is the special case of $L_w^{p),\varphi(x)}(\Omega)$ taking $\varphi(x) = \frac{x^\theta}{|\Omega|}$.

Throughout the paper the symbol $\varphi(t) \approx \psi(t)$ means that there exist positive constants c_1 and c_2 such that $c_1\varphi(t) \leq \psi(t) \leq c_2\psi(t)$. Constants (often different constants in the same series of inequalities) will generally be denoted by c or C. By the symbol p' we denote the conjugate number of p, i.e. $p' := \frac{p}{p-1}$, $1 < p < \infty$.

2 Fractional Integrals and Fractional Maximal Functions in Unweighted Grand Lebesgue Spaces

Let

$$(I_\alpha f)(x) = \int_0^1 \frac{f(y)}{|x-y|^{1-\alpha}}\, dy, \quad 0 < \alpha < 1$$

be the Riesz potential operator defined on $[0, 1]$. We begin with the following result:

Theorem 2.1. *Let $0 < \alpha < 1$, $1 < p < \frac{1}{\alpha}$, θ_1 and θ_2 be positive numbers such that $\theta_2 < \theta_1(1 + \alpha q)$, where $q = \frac{p}{1-\alpha p}$. Then the operator I_α is not bounded from $L^{p),\theta_1}$ to $L^{q),\theta_2}$.*

Proof. Suppose the contrary: I_α is bounded from $L^{p),\theta_1}$ to $L^{q),\theta_2}$ i. e. the inequality

$$\|I_\alpha f\|_{L^{q),\theta_2}} \leq c\|f\|_{L^{p),\theta_1}} \tag{2.1}$$

holds, where the positive constant c does not depend on f. Taking $f = \chi_J$ in (2.1), where J is an interval in $[0, 1]$, we have

$$(I_\alpha f)(x) = \int_J \frac{dy}{|x-y|^{1-\alpha}} \geq |J|^\alpha, \;\; x \in J.$$

Consequently,

$$\|I_\alpha f\|_{L^{q),\theta_2}} \geq |J|^\alpha \|\chi_J\|_{L^{q),\theta_2}}.$$

Taking inequality (2.1) into account we have that

$$|J|^\alpha \|\chi_J\|_{L^{q),\theta_2}} \leq c\|\chi_J\|_{L^{p),\theta_1}}, \tag{2.2}$$

where the positive constant c does not depend on J.

Let us define the number ε_J which is between 0 and $p-1$ and satisfies the condition

$$\sup_{0<\varepsilon\le p-1}\left(\varepsilon^{\theta_1}|J|\right)^{\frac{1}{p-\varepsilon}}=\left(\varepsilon_J^{\theta_1}|J|\right)^{\frac{1}{p-\varepsilon_J}} \tag{2.3}$$

Now we claim that $\lim\limits_{|J|\to 0}\varepsilon_J=0$. Indeed, suppose the contrary: that there is a sequence of intervals J_n and a positive number λ such that $|J_n|\to 0$ and $\varepsilon_{J_n}\ge\lambda>0$ for all $n\in N$. It is obvious that we can choose J_{n_0} so that

$$\frac{|J_{n_0}|^{\frac{1}{\theta_1}}(p-1)}{e}<e^{-\frac{p}{\lambda/2}}.$$

Now we claim that $f'(x)<0$ for all $x\in[\lambda/2,p-1]$, where $f(x)=\left(x^{\theta_1}|J_{n_0}|\right)^{\frac{1}{p-x}}$. Indeed, it is easy to see that for $\lambda/2\le x\le p-1$, the inequalities

$$\frac{|J_{n_0}|^{\frac{1}{\theta_1}}x}{e}\le\frac{|J_{n_0}|^{\frac{1}{\theta_1}}(p-1)}{e}<e^{-\frac{p}{\lambda/2}}\le e^{-\frac{p}{x}}.$$

hold. Hence, using the formula

$$f'(x)=f(x)\cdot\frac{1}{p-x}\left[\frac{ln\left(x^{\theta_1}|J_{n_0}|\right)}{p-x}+\frac{\theta_1}{x}\right]$$

and the fact that

$$f'(x)<0\Longleftrightarrow\frac{x|J_{n_0}|^{\frac{1}{\theta_1}}}{e}<e^{-\frac{p}{x}}$$

we conclude that $f'(x)<0$.

This observation together with the equality $\lim\limits_{x\to 0}f(x)=0$ gives that $\varepsilon_{J_{n_0}}<\lambda$, where $\varepsilon_{J_{n_0}}$ is defined by

$$\sup_{0<\varepsilon\le p-1}\left(\varepsilon^{\theta_1}|J_{n_0}|\right)^{\frac{1}{p-\varepsilon}}=\left(\varepsilon_{J_{n_0}}^{\theta_1}|J_{n_0}|\right)^{1/(p-\varepsilon_{J_{n_0}})}.$$

This contradicts the assumption that $\varepsilon_{J_n}\ge\lambda>0$ for all n. Further, we choose η_J so that

$$\alpha=\frac{1}{p}-\frac{1}{q}=\frac{1}{p-\varepsilon_J}-\frac{1}{q-\eta_J}.$$

This is equivalent to say that

$$\eta_J=q-\frac{p-\varepsilon_J}{1-\alpha(p-\varepsilon_J)}. \tag{2.4}$$

By (2.2) and (2.3) we have that

$$|J|^{\alpha}\eta_J^{\frac{\theta_2}{q-\eta_J}}|J|^{\frac{1}{q-\eta_J}}\le c\varepsilon_J^{\frac{\theta_1}{p-\varepsilon_J}}|J|^{\frac{1}{p-\varepsilon_J}}. \tag{2.5}$$

(here we used the fact that if ε_J is small, then $0 < \eta_J < q-1$). Now (2.5) yield:

$$\eta_J^{\frac{\theta_2}{q-\eta_J}} \varepsilon_J^{-\frac{\theta_1}{p-\varepsilon_J}} \le c. \tag{2.6}$$

Further, (2.4) and (2.6) imply

$$\left(\frac{q - \frac{p-\varepsilon_J}{1-\alpha(p-\varepsilon_J)}}{\varepsilon_J} \right)^{\frac{\theta_2}{p-\varepsilon_J}-\alpha\theta_2} \varepsilon_J^{-\frac{\theta_1}{p-\varepsilon_J}+\frac{\theta_2}{p-\varepsilon_J}-\alpha\theta_2} \le c. \tag{2.7}$$

Passing now to the limit as $|J| \to 0$ we see that the left-hand side of (2.7) tends to $+\infty$ because the limit of the first factor is $\left[\frac{1}{(1-\alpha p)^2}\right]^{\frac{\theta_2}{p}-\alpha\theta_2}$, and

$$\lim_{|J|\to 0} \varepsilon_J^{\frac{\theta_2-\theta_1}{p-\varepsilon_J}-\alpha\theta_2} = \lim_{|J|\to 0} \varepsilon_J^{\frac{\theta_2-\theta_1}{p}-\alpha\theta_2} = \infty$$

(Here we used the observation $\frac{\theta_2}{\theta_1} < 1 + \alpha q \Longleftrightarrow \frac{\theta_2-\theta_1}{p} - \alpha\theta_2 < 0$). $\square$

Analysing the proof of Theorem 2.1 we have the result similar to that of the previous statement for the fractional maximal operator

$$M_\alpha f(x) = \sup_{\substack{J \ni x \\ J \subset [0,1]}} \frac{1}{|J|^{1-\alpha}} \int_J |f|, \quad x \in [0,1].$$

Theorem 2.2. *Let the conditions of Theorem* 2.1 *be satisfied. Then the operator* M_α *is not bounded from* $L^{p),\theta_1}$ *to* $L^{q),\theta_2}$.

Proof. Proof is the same as in the case of Theorem 2.1. We only need to observe that the inequality

$$M_\alpha f(x) \ge \frac{1}{|J|^{1-\alpha}} \int_J dx = |J|^\alpha, \ \ x \in J,$$

holds for $f(x) = \chi_J(x)$, where J is a subinterval of $[0,1]$. Details are omitted. $\square$

3 Sobolev's Embedding in Weighted Generalized Grand Lebesgue Spaces

This section is devoted to the investigation of the one–weight inequality for the operator I_α in $L^{p),\theta}_w$ spaces.

First we introduce the function

$$\varphi(u) = \left[\frac{u-q}{1-\alpha(u-q)} + p \right]^{1-(u-q)\alpha} \tag{3.1}$$

where $0<\alpha<1$, $1<p<\frac{1}{\alpha}$, $q=\frac{p}{1-\alpha p}$.

To prove the main results we need some auxiliary statements.

Lemma 3.1. *$\varphi(x)\approx x^{1+\alpha q}$ near 0.*

The proof is straightforward and therefore is omitted.

Lemma 3.2. *Let $1<q<\infty$ and let w be a weight. Then*

$$\|f\|_{L_w^{q),\varphi(x)}([0,1])}\approx\|f\|_{L_w^{q),1+\alpha q}([0,1])}$$

where φ is defined by (3.1).

Proof. Follows immediately from Lemma 3.1. $\square$

Lemma 3.3. *Let $1<q<\infty$ and let $\theta>0$. Then*

$$\|f\|_{L_w^{q),\varphi(x^\theta)}([0,1])}\approx\|f\|_{L_w^{q),\theta(1+\alpha q)}([0,1])},$$

where φ is defined by (3.1)

The proof follows immediately from Lemma 3.1.

Lemma 3.4. *Let $1<p<\infty$ and let φ be as above. Then there is a positive constant c such that for all intervals $J\subset[0,1]$ and $f\in L_w^{p,\varphi(x)}$ the inequality*

$$\|f\|_{L_w^{p),\varphi(x)}(J)}\le c(w(J))^{-\frac{1}{p}}\left(\int\limits_J|f(t)|^p w(t)dt\right)^{\frac{1}{p}}\|\chi_J\|_{L_w^{p),\varphi(x)}}$$

holds.

Proof. We have

$$\|f\|_{L_w^{p),\varphi(x)}(J)}=\sup_{0<\varepsilon\le p-1}\left(\varphi(\varepsilon)\int\limits_J|f(x)|^{p-\varepsilon}w(x)dx\right)^{\frac{1}{p-\varepsilon}}$$

$$=\sup_{0<\varepsilon\le p-1}\left(\varphi(\varepsilon)\int\limits_J|f(x)|^{p-\varepsilon}w(x)^{\frac{p-\varepsilon}{p}}w(x)^{\frac{\varepsilon}{p}}dx\right)^{\frac{1}{p-\varepsilon}}$$

$$\le\sup_{0<\varepsilon\le p-1}\varphi(\varepsilon)^{\frac{1}{p-\varepsilon}}\left(\int\limits_J\left(|f(x)|^{p-\varepsilon}w(x)^{\frac{p-\varepsilon}{p}}\right)^{\frac{p}{p-\varepsilon}}dx\right)^{\frac{1}{p}}\left(\int\limits_J\left[w^{\frac{\varepsilon}{p}}(x)\right]^{\frac{p}{\varepsilon}}dx\right)^{\frac{\varepsilon}{p(p-\varepsilon)}}$$

$$= \sup_{0<\varepsilon\le p-1} \varphi(\varepsilon)^{\frac{1}{p-\varepsilon}} \left(\int_J |f(x)|^p w(x)dx \right)^{\frac{1}{p}} \left(\int_J w(x)dx \right)^{\frac{\varepsilon}{p(p-\varepsilon)}}$$

$$= \left(\int_J |f(x)|^p w(x)dx \right)^{\frac{1}{p}} \left(\int_J w(x)dx \right)^{-\frac{1}{p}} \sup_{0<\varepsilon\le p-1} \left(\varphi(\varepsilon) \int_J w(x)dx \right)^{\frac{1}{p-\varepsilon}}$$

$$= \left(\int_J |f(x)|^p w(x)dx \right)^{\frac{1}{p}} (w(J))^{-\frac{1}{p}} \|\chi_J\|_{L_w^{p),\varphi(x)}(J)}.$$

$\square$

Lemma 3.5. *Let $\theta > 0$, $1 < p < \infty$, $0 < \alpha < 1/p$ and let $q = \frac{p}{1-\alpha p}$. Suppose that the inequality*

$$\|I_\alpha(fw^\alpha)\|_{L_w^{q),\theta}([0,1])} \le c\|f\|_{L_w^{p),\theta}([0,1])} \tag{3.2}$$

holds. Then

$$\int_0^1 w^{-p'/q}(x)dx < \infty.$$

Proof. Suppose the contrary: $\int_0^1 w^{-p'/q}(x)dx = \|w^{\alpha-1}\|_{L_w^{p'}} = \infty$. This means that there is a function $g \in L_w^p$ such that $\int_0^1 gw^\alpha = \infty$.

On the other hand,

$$I_\alpha(gw^\alpha)(x) = \int_0^1 \frac{g(t)w^\alpha(t)}{|x-t|^{1-\alpha}}dt \ge \int_0^1 g(t)w^\alpha(t)dt = \infty, \quad x \in [0,1].$$

Further, Lemma 3.4 with $\varphi(x) = x^\theta$ implies that $g \in L_w^{p),\theta}([0,1])$. But $I_\alpha(gw^\alpha)(x) = \infty$ for $x \in [0,1]$. This contradicts inequality (3.2). $\square$

Definition 3.1. Let $1 < r < \infty$. We say that a weight function w belongs to the Muckenhoupt's class $A_r([0,1])$ ($w \in A_r([0,1])$) if

$$A_r(w) := \sup_{J\subset[0,1]} \left(\frac{1}{|J|} \int_J w \right)^{1/r} \left(\frac{1}{|J|} \int_J w^{1-r'} \right)^{1/r'} < \infty,$$

where the supremum is taken over all subintervals J of $[0,1]$.

Lemma 3.6. *Let $0<\alpha<1$, $1<p<1/\alpha$. We set $q=\frac{p}{1-\alpha p}$. Suppose that $w\in A_{1+q/p'}([0,1])$, i.e.,*

$$\sup_{J\subset[0,1]}\left(\frac{1}{|J|}\int_J w\right)^{1/q}\left(\frac{1}{|J|}\int_J w^{-p'/q}\right)^{1/p'}<\infty.$$

Then there are positive constants σ_1, σ_2 and L satisfying the conditions:

$$\frac{1}{p-\sigma_2}-\frac{1}{q-\sigma_1}=\alpha,\quad w\in A_{1+\frac{q-\sigma_1}{(p-\sigma_2)'}},$$
$$\|K_\alpha\|_{L^{p-\eta}_w\to L^{q-\varepsilon}_w}\le L$$

for all $0\le\varepsilon\le\sigma_1$, $0\le\eta\le\sigma_2$ with $\frac{1}{p-\eta}-\frac{1}{q-\varepsilon}=\alpha$, where K_α is the operator defined as follows $K_\alpha f=I_\alpha(fw^\alpha)$.

Proof. Since $w\in A_{1+q/p'}$ by the openness property of Muckenhoupt's classes (see [9]) we have that there are small positive numbers σ_1 and σ_2 such that $\frac{1}{p-\sigma_2}-\frac{1}{q-\sigma_1}=\alpha$ and $w\in A_{1+(q-\sigma_1)/(p-\sigma_2)'}$.

By the result of B. Muckenhoupt and R. L. Wheeden [10] we have that the operator K_α is bounded from L^p_w to L^q_w and from $L^{p-\sigma_2}_w$ to $L^{q-\sigma_1}_w$. Let $0<t<1$ and let us define positive numbers η and ε so that

$$\frac{1}{p-\eta}=\frac{t}{p}+\frac{1-t}{p-\sigma_2},\qquad\frac{1}{q-\varepsilon}=\frac{t}{q}+\frac{1-t}{q-\sigma_1}.$$

Then by applying the Rieasz–Thorin theorem (see e.g. [2], p. 16) we have that K_α is bounded from $L^{p-\eta}$ to $L^{q-\varepsilon}$ and moreover,

$$\|K_\alpha\|_{L^{p-\eta}_w\to L^{q-\varepsilon}_w}\le\|K_\alpha\|^t_{L^p_w\to L^q_w}\|K_\alpha\|^{1-t}_{L^{p-\sigma_2}_w\to L^{q-\sigma_1}_w}.$$

Observe now that

$$\frac{1}{p-\eta}-\frac{1}{q-\varepsilon}=\frac{t}{p}-\frac{t}{q}+\frac{1-t}{p-\sigma_2}-\frac{1-t}{q-\sigma_1}$$

$$=t\Big(\frac{1}{p}-\frac{1}{q}\Big)+(1-t)\Big(\frac{1}{p-\sigma_2}-\frac{1}{q-\sigma_1}\Big)=t\alpha+(1-t)\alpha=\alpha.$$

The lemma is proved since we can take $L=\|K_\alpha\|_{L^p_w\to L^q_w}\|K_\alpha\|_{L^{p-\sigma_2}_w\to L^{q-\sigma_1}_w}$ (since without loss of generality we can assume that each term is greater or equal to 1). $\square$

Theorem 3.1. *Let $1<p<\infty$ and let $0<\alpha<1/p$. Suppose that $\theta>0$. We set $q=\frac{p}{1-\alpha p}$. Then the inequality*

$$\|I_\alpha(fw^\alpha)\|_{L^{q),\theta(1+\alpha q)}_w([0,1])}\le c\|f\|_{L^{p),\theta}_w([0,1])}\tag{3.3}$$

holds if and only if $w\in A_{1+q/p'}([0,1])$.

Proof. By Lemma 3.1 we have that (3.3) is equivalent to the inequality

$$\|I_\alpha(fw^\alpha)\|_{L_w^{q),\psi(x)}([0,1])} \le c\|f\|_{L_w^{p),\theta}([0,1])}, \tag{3.4}$$

where

$$\psi(x) = \varphi(x^\theta), \quad \varphi(x) = \left[\frac{x-q}{1-\alpha(x-q)} + p\right]^{1-(x-1)\alpha}. \tag{3.5}$$

Necessity. Let (3.3) and hence (3.4) hold. By Lemma 3.5 we have that $\int\limits_0^1 w^{-p'/q} < \infty$. Let us take $f = \chi_J w^{-\alpha-p'/q}$. Then for $x \in J$, we get that

$$I_\alpha(w^\alpha f)(x) \ge \frac{1}{|J|^{1-\alpha}} \int\limits_J w^\alpha f = \frac{1}{|J|^{1-\alpha}} \int\limits_J w^{-p'/q}.$$

Hence,

$$\|I_\alpha(w^\alpha f)\|_{L_w^{q),\psi(x)}([0,1])} \ge |J|^{\alpha-1} \left(\int\limits_J w^{-p'/q}\right) \|\chi_J\|_{L_w^{q),\psi(x)}([0,1])}.$$

Further, by Lemma 3.4 we find that

$$|J|^{\alpha-1} \left(\int\limits_J w^{-p'/q}\right) \|\chi_J\|_{L_w^{q),\psi(x)}([0,1])}$$

$$\le c\|f\|_{L^{p),\theta}([0,1])} \le c(w(J))^{-\frac{1}{p}} \left(\int\limits_J |f(t)|^p w(t)dt\right)^{\frac{1}{p}} \|\chi_J\|_{L_w^{p),\theta}([0,1])}$$

$$= cw(J)^{-\frac{1}{p}} \left(\int\limits_J w^{-p'/q}\right)^{1/p} \|\chi_J\|_{L_w^{p),\theta}([0,1])}.$$

Further, it is easy to see that there is a number η_J depending on J such that $0 < \eta_J \le p-1$ and

$$|J|^{\alpha-1} w(J)^{\frac{1}{p}} \left(\int\limits_J w^{-p'/q}\right)^{\frac{1}{p'}} \|\chi_J\|_{L_w^{q),\psi(x)}([0,1])} \le c\left(\eta_J w(J)\right)^{\frac{1}{p-\eta_J}}.$$

For such η_J we choose ε_J so that

$$\frac{1}{p-\eta_J} - \frac{1}{q-\varepsilon_J} = \alpha.$$

Then $0 < \varepsilon_J \le q-1$ and

$$|J|^{\alpha-1} w(J)^{\frac{1}{p}-\frac{1}{p-\eta_J}} \eta_J^{-\frac{\theta}{p-\eta_J}} \psi(\varepsilon_J)^{\frac{1}{q-\varepsilon_J}} w(J)^{\frac{1}{q-\varepsilon_J}} \left(\int_J w^{-p'/q} \right)^{\frac{1}{p'}} \le c.$$

Observe that by Lemma 3.1 we have that

$$\eta_J^{-\frac{\theta}{p-\eta_J}} \psi(\varepsilon_J)^{\frac{1}{q-\varepsilon_J}} = \eta_J^{-\frac{\theta}{p-\eta_J}} \varphi\left(\varepsilon_J^{\theta}\right)^{\frac{1}{q-\varepsilon_J}} \approx \eta_J^{-\frac{\theta}{p-\eta_J}} \varepsilon_J^{\frac{\theta(1+\alpha q)}{q-\varepsilon_J}} = \left(\eta_J^{-\frac{1}{p-\eta_J}} \varepsilon_J^{\frac{1+\alpha q}{q-\varepsilon_J}} \right)^{\theta}$$

$$\approx \left(\eta_J^{-\frac{1}{p-\eta_J}} \varphi(\varepsilon_J)^{\frac{1}{q-\varepsilon_J}} \right)^{\theta} = 1$$

and also,

$$\frac{1}{p} - \frac{1}{p-\eta_J} + \frac{1}{q-\varepsilon_J} = \frac{1}{p} - \alpha = \frac{1}{q}.$$

Finally, we have that

$$|J|^{\alpha-1} w(J)^{\frac{1}{q}} \left(\int_J w^{-p'/q} \right)^{1/p'} \le c.$$

Necessity is proved.

Sufficiency. Using Lemma 3.6 we have that there are positive constants σ_1, σ_2 and L satisfying the conditions: $\frac{1}{p-\sigma_2} - \frac{1}{q-\sigma_1} = \alpha$, $w \in A_{1+\frac{q-\sigma_1}{(p-\sigma_2)'}}$, $\|K_\alpha\|_{L_w^{p-\eta} \to L_w^{q-\varepsilon}} \le L$ for all $0 \le \varepsilon \le \sigma_1$, $0 \le \eta \le \sigma_2$ with $\frac{1}{p-\eta} - \frac{1}{q-\varepsilon} = \alpha$, where K_α is the operator defined by $K_\alpha f = I_\alpha(fw^\alpha)$.

Let σ be a small positive number such that $\sigma < \sigma_1 < q-1$ and let us fix $\varepsilon \in (\sigma, q-1]$. Then $\frac{q-\sigma}{q-\varepsilon} > 1$. By Hölder's inequality we have that

$$\|I_\alpha(fw^\alpha)\|_{L_w^{q-\varepsilon}([0,1])} \le \left(\int_0^1 |I_\alpha(fw^\alpha)(x)|^{q-\sigma} w(x) dx \right)^{\frac{1}{q-\sigma}} w([0,1])^{\frac{\varepsilon-\sigma}{(q-\sigma)(q-\varepsilon)}}$$

because $\left(\frac{q-\sigma}{q-\varepsilon}\right)' = \frac{q-\sigma}{\varepsilon-\sigma}$.

Further, the conditions $\sigma < q-1$ and $\sigma < \varepsilon < q-1$ yield

$$0 < \frac{\varepsilon-\sigma}{(q-\sigma)(q-\varepsilon)} < \frac{q-1-\sigma}{q-\sigma}, \quad (q-1)\sigma^{-\frac{1}{q-\sigma}} > 1.$$

Consequently, using the well–known result by B. Muckenhoupt and R. L. Wheeden [10] for the classical weighted Lebesgue spaces:

$$\|I_\alpha(fw^\alpha)\|_{L_w^q([0,1])} \le c\|f\|_{L_w^p([0,1])} \Longleftrightarrow w \in A_{1+q/p'}([0,1]), \quad q = \frac{p}{1-\alpha p},$$

we find that

$$\|I_\alpha f\|_{L^{q),\psi(x)}_w([0,1])} = \max\left\{ \sup_{0<\varepsilon\le\sigma} \psi(\varepsilon)^{\frac{1}{q-\varepsilon}} \|I_\alpha(fw^\alpha)\|_{L^{q-\varepsilon}_w([0,1])}, \right.$$

$$\left. \sup_{\sigma<\varepsilon\le q-1} \psi(\varepsilon)^{\frac{1}{q-\varepsilon}} \|I_\alpha(fw^\alpha)\|_{L^{q-\varepsilon}_w([0,1])} \right\}$$

$$\le \max\left\{ \sup_{0<\varepsilon\le\sigma} \psi(\varepsilon)^{\frac{1}{q-\varepsilon}} \|I_\alpha(fw^\alpha)\|_{L^{q-\varepsilon}_w([0,1])}, \right.$$

$$\left. \sup_{\sigma<\varepsilon\le q-1} \psi(\varepsilon)^{\frac{1}{q-\varepsilon}} \|I_\alpha(fw^\alpha)\|_{L^{q-\varepsilon}_w} w([0,1])^{\frac{\varepsilon-\sigma}{(q-\sigma)(q-\varepsilon)}} \right\}$$

$$\le \max\left\{1, \sup_{\sigma<\varepsilon\le q-1} \psi(\varepsilon)^{\frac{1}{q-\varepsilon}} \psi(\sigma)^{-\frac{1}{q-\sigma}} w([0,1])^{\frac{\varepsilon-\sigma}{(q-\sigma)(q-\varepsilon)}} \right\} \sup_{0<\varepsilon\le\sigma} \psi(\varepsilon)^{\frac{1}{q-\varepsilon}} \|I_\alpha(fw^\alpha)\|_{L^{q-\varepsilon}_w([0,1])}$$

$$\le c\max\left\{1, \left[\sup_{\sigma<\varepsilon\le q-1} (\psi(\varepsilon))^{\frac{1}{q-\varepsilon}}\right] \varphi(\sigma)^{-\frac{1}{q-\sigma}} (1+w([0,1])^{\frac{q-1-\sigma}{q-\sigma}} \right\} \sup_{0<\eta\le\sigma_0} \eta^{\frac{\theta}{p-\eta}} \|f\|_{L^{p-\eta}_w([0,1])}$$

$$\le c\left(\sup_{\sigma<\varepsilon\le q-1} \psi(\varepsilon)^{\frac{1}{q-\varepsilon}}\right) \varphi(\sigma)^{-\frac{1}{q-\sigma}} (1+w([0,1]))^{\frac{q-1-\sigma}{q-\sigma}} \|f\|_{L^{p),\theta}_w([0,1])}.$$

Here σ_0 is small positive number such that when $0<\varepsilon\le\sigma$, then $0<\eta\le\sigma_0<\sigma_1<p-1$. Also, we used the estimates:

$$\psi(\varepsilon)^{\frac{1}{q-\varepsilon}} \approx \varepsilon^{\frac{\theta(1+\alpha q)}{q-\varepsilon}} \approx \varphi(\varepsilon)^{\frac{\theta}{q-\varepsilon}} = \eta^{\frac{\theta}{p-\eta}}, \quad \text{as} \quad \varepsilon\to 0,$$

where $\frac{1}{p-\eta} - \frac{1}{q-\varepsilon} = \alpha$. □

Corollary 3.1. *Let $\theta>0$ and let $1<p<\infty$. Suppose that $0<\alpha<1/p$. We set $q=\frac{p}{1-\alpha p}$. Then I_α is bounded from $L^{p),\theta_1}([0,1])$ to $L^{q),\theta_2}([0,1])$ provided that $\theta_2>(1+\alpha q)\theta_1$.*

Proof follows immediately from Theorem 3.1 (in the unweighted case $w(x)\equiv$ *const*) and (1.1). □

4 One-sided potentials

In this section we show that the unboudedness result in grand Lebesgue spaces is also true for the one–sided potentials:

$$(R_\alpha f)(x) = \int_0^x \frac{f(t)}{(x-t)^{1-\alpha}} dt, \quad x \in (0,1);$$

and

$$(W_\alpha f)(x) = \int_x^1 \frac{f(t)}{(t-x)^{1-\alpha}} dt, \quad x \in (0,1),$$

where $0 < \alpha < 1$. In particular, we claim that R_α and W_α are not bounded from $L^{p),\theta_1}$ to $L^{q),\theta_2}$, where $q = \frac{p}{1-\alpha p}$, $1 < p < \infty$, θ_1, $\theta_2 > 0$, $\theta_2 < \frac{\theta_1 q}{p}$. Indeed, let us show the result first for R_α.

Suppose the contrary:

$$\|R_\alpha f\|_{L^{q),\theta_2}([0,1])} \leq c\|f\|_{L^{p),\theta_1}([0,1])}, \qquad \theta_2 < \frac{\theta_1 q}{p}, \tag{4.1}$$

where c does not depend on f. Let $f_n(x) = \chi_{(0,1/2n)}(x)$ in (4.1). Then taking the following inequality

$$(R_\alpha f_n)(x) \geq \int_0^{\frac{1}{2n}} \frac{1}{(x-t)^{1-\alpha}} dt \geq (\frac{1}{2n})^\alpha, \quad x \in \left(\frac{1}{2n}, \frac{1}{n}\right), \tag{4.2}$$

into account, (4.1) yields that

$$(2n)^{-\alpha} \left\| \chi_{(\frac{1}{2n}, \frac{1}{n})} \right\|_{L^{q),\theta_2}([0,1])} \leq c \left\| \chi_{(0,1/2n)} \right\|_{L^{p),\theta_1}([0,1])}. \tag{4.3}$$

Now we choose ε_n positive number so that

$$\sup_{0<\varepsilon\leq p-1} \left(\varepsilon^{\theta_1} \frac{1}{2n} \right)^{\frac{1}{p-\varepsilon}} = \left(\varepsilon_n^{\theta_1} \frac{1}{2n} \right)^{\frac{1}{p-\varepsilon_n}}. \tag{4.4}$$

We now observe that $\lim_{n\to 0} \varepsilon_n = 0$ (see the proof of Theorem 2.1 for the similar arguments). Choose now η_n so that

$$\alpha = \frac{1}{p} - \frac{1}{q} = \frac{1}{p-\varepsilon_n} - \frac{1}{q-\eta_n}.$$

Hence,

$$\eta_n = q - \frac{p-\varepsilon_n}{1-\alpha(p-\varepsilon_n)}. \tag{4.5}$$

By (4.3)-(4.5) we conclude that

$$(2n)^{-\alpha} \eta_n^{\frac{\theta_2}{q-\eta_n}} \left(\frac{1}{2n}\right)^{\frac{1}{q-\eta_n}} \leq c\varepsilon_n^{\frac{\theta_1}{p-\varepsilon_n}} (2n)^{-1/(p-\varepsilon_n)}. \tag{4.6}$$

From (4.6) we have that

$$\eta_n^{\frac{\theta_2}{q-\eta_n}} \varepsilon_n^{-\frac{\theta_1}{p-\varepsilon_n}} \leq c_p, \quad \text{for all } n \in N \tag{4.7}$$

because

$$\frac{1}{2} \leq \left(\frac{1}{2}\right)^{\frac{1}{p-\varepsilon_n}} \leq \left(\frac{1}{2}\right)^{\frac{1}{p}},$$

$$\frac{1}{2} \leq \left(\frac{1}{2}\right)^{\frac{1}{q-\eta_n}} \leq \left(\frac{1}{2}\right)^{\frac{1}{q}}.$$

Now (4.5) yields

$$\left[\frac{q - \frac{p-\varepsilon_n}{1-\alpha(p-\varepsilon_n)}}{\varepsilon_n}\right]^{\frac{\theta_2}{p-\varepsilon_n}-\alpha\theta_2} \cdot \varepsilon_n^{-\frac{\theta_1}{p-\varepsilon_n}+\frac{\theta_2}{p-\varepsilon_n}-\alpha\theta_2} \leq c_p.$$

Hence,

$$\left[\frac{q - \frac{p-\varepsilon_n}{1-\alpha(p-\varepsilon_n)}}{\varepsilon_n}\right]^{\frac{\theta_2}{p-\varepsilon_n}-\alpha\theta_2} \varepsilon_n^{\frac{\theta_2-\theta_1}{p-\varepsilon_n}-\alpha\theta_2} \leq c_p,$$

which is impossible, because $\lim\limits_{n\to\infty} \varepsilon_n^{\frac{\theta_2-\theta_1}{p-\varepsilon_n}-\alpha\theta_2} = \infty$ (recall that $\frac{\theta_2-\theta_1}{p} - \alpha\theta_2 = \frac{\theta_2}{q} - \frac{\theta_1}{p} < 0$).

Analogously, we have that W_α is not bounded from $L^{p),\theta_1}$ to $L^{q),\theta_2}$. This follows from the inequalities

$$(W_\alpha)(x) \geq \int\limits_x^{1-\frac{1}{3n}} \frac{f(t)}{(t-x)^{1-\alpha}} dt \geq \left(\frac{2}{3n}\right)^{\alpha-1} \cdot \frac{1}{6n} = c_\alpha n^{-\alpha}, \quad x \in \left(1-\frac{1}{n}, 1-\frac{1}{2n}\right),$$

where $f(t) = \chi_{(1-\frac{1}{2n}, 1-\frac{1}{3n})}(t)$. Hence,

$$c_\alpha n^{-\alpha} \left\|\chi_{\left(1-\frac{1}{n}, 1-\frac{1}{2n}\right)}\right\|_{L^{q),\theta_2}([0,1])} \leq c \left\|\chi_{\left(1-\frac{1}{2n}, 1-\frac{1}{3n}\right)}\right\|_{L^{p),\theta_1}([0,1])}.$$

Choosing now ε_n so that

$$\left[\varepsilon_n^{\theta_1} \frac{1}{6n}\right]^{\frac{1}{p-\varepsilon_n}} = \sup_{0<\varepsilon_n \leq p-1} \left[\varepsilon_n^{\theta_1} \frac{1}{6n}\right]^{\frac{1}{p-\varepsilon}}, \quad 0 < \varepsilon_n \leq p-1,$$

and observing that $\lim\limits_{n\to\infty} \varepsilon_n = 0$ (see the proof of Theorem 2.1 for the similar arguments) we find that the conclusion similar to the case of R_α is valid.

4.1 Conclusions and Remarks

Let $0 < \alpha < 1$ and let I_α, R_α, W_α be potential operators defined above. In the sequel we denote by T_α one of these operators.

Corollary 5.1. *Let $1 < p < \infty$ and let $0 < \alpha < 1/p$. We set $q = \frac{p}{1-\alpha p}$. Suppose that θ_1 and θ_2 be positive numbers. Then:*

(i) *If $\theta_2 < (1 + \alpha q)\theta_1$, then T_α is not bounded from $L^{p),\theta_1}$ to $L^{q),\theta_2}$.*

(ii) *If $\theta_2 \geq (1 + \alpha q)\theta_1$, then T_α is bounded from $L^{p),\theta_1}$ to $L^{q),\theta_2}$.*

Remark 5.1. There is a function f from $L^{p)} \backslash L^p$ such that $T_\alpha f \in L^{q)} \backslash L^q$.

Indeed, let $f(t) = t^{-\frac{1}{p}}$, $t \in (0, 1)$. Then $f \in L^{p)} \backslash L^p$. On the other hand, (see e. g. [11]), $T_\alpha f \approx t^{-\frac{1}{q}}$. Hence $T_\alpha f \in L^{q)} \backslash L^q$.

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